

# Decoherence and asymptotic entanglement in open quantum dynamics

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## Abstract

In the framework of the theory of open systems based on completely positive quantum dynamical semigroups, we determine the degree of quantum decoherence of a harmonic oscillator interacting with a thermal bath. It is found that the system manifests a quantum decoherence which is more and more significant in time. We also calculate the decoherence time and show that it has the same scale as the time after which thermal fluctuations become comparable with quantum fluctuations. Then we solve the master equation for two independent harmonic oscillators interacting with an environment in the asymptotic long-time regime. We give a description of the continuous-variable asymptotic entanglement in terms of the covariance matrix of the quantum states of the considered system for an arbitrary Gaussian input state. Using the Peres–Simon necessary and sufficient condition for separability of two-mode Gaussian states, we show that the two non-interacting systems immersed in a common environment become asymptotically entangled for certain environments, so that in the long-time regime they manifest non-local quantum correlations.

*Key words:* Open systems, quantum decoherence, quantum entanglement, inseparable states.

## 1 Introduction

By quantum decoherence (QD) [1, 2] we understand the irreversible, uncontrollable and persistent formation of quantum correlations (entanglement) of a system with its environment [3], expressed by the damping of the coherences present in the quantum state of the system, when the off-diagonal elements of the density matrix decay below a certain level, so that this density matrix becomes approximately diagonal.

Since QD strongly depends on the interaction between the system and its external environment [1, 2], its role became relevant in many interesting physical problems. In many cases one is interested in understanding QD because one wants to prevent decoherence from damaging quantum states and to protect the information stored in

quantum states from the degrading effect of the interaction with the environment. QD is also responsible for washing out the quantum interference effects which are desirable to be seen as signals in experiments and it has a negative influence on many areas relying upon quantum coherence effects, in particular QD is a major problem in quantum optics and physics of quantum information and computation [4].

Quantum entanglement represents the physical resource in quantum information science which is indispensable for the description and performance of such tasks like teleportation, superdense coding, quantum cryptography and quantum computation [4]. Therefore the generation, detection and manipulation of the entanglement continues to be presently a problem of intense investigation.

When two systems are immersed in an environment, then, besides and at the same time with the QD phenomenon, the external environment can also generate a quantum entanglement of the two systems and therefore an additional mechanism to correlate them [5, 6, 7]. In certain circumstances, the environment enhances entanglement and in others it suppresses the entanglement and the state describing the two systems becomes separable. The structure and properties of the environment may be such that not only the two systems become entangled, but also such that a certain amount of entanglement survives in the asymptotic long-time regime. The reason is that even if not directly coupled, the two systems immersed in the same environment can interact through the environment itself and it depends on how strong this indirect interaction is with respect to the QD, whether entanglement can be generated at the beginning of the evolution and, in the case of an affirmative answer, if it can be maintained for a definite time or it survives indefinitely in time [5].

In this work we study QD of a harmonic oscillator interacting with an environment, in particular with a thermal bath, in the framework of the theory of open quantum systems based on completely positive dynamical semigroups. We determine the degree of QD [8] for different regimes of the temperature of environment. It is found that the system manifests a QD which in general increases with time and temperature. We also calculate the decoherence time and show that it has the same scale as the time after which thermal fluctuations become comparable with quantum fluctuations. Then we investigate, in the same framework, the existence of the continuous variable asymptotic entanglement for a subsystem composed of two identical harmonic oscillators interacting with an environment. We are interested in discussing the correlation effect of the environment, therefore we assume that the two systems are independent, i.e. they do not interact directly. The initial state of the subsystem

is taken of Gaussian form and the evolution under the quantum dynamical semigroup assures the preservation in time of the Gaussian form of the state. We only investigate here the asymptotic behaviour of the subsystem states. The time evolution of the entanglement, in particular the possibility of the so-called "entanglement sudden death", that is suppression of the entanglement at a certain finite moment of time, will be discussed in a future work.

The organizing of the paper is as follows. In Sec. 2 we review the Markovian master equation for the damped harmonic oscillator and solve it in coordinate representation. Then in Sec. 3 we investigate QD and calculate the decoherence time of the system. In Sec. 4 we write the equations of motion in the Heisenberg picture for two independent harmonic oscillators interacting with a general environment. With these equations we derive in Sec. 5 the asymptotic values of the variances and covariances of the coordinates and momenta which enter the asymptotic covariance matrix. Then, by using the Peres-Simon necessary and sufficient condition for separability of two-mode Gaussian states [9, 10], we investigate the behaviour of the environment induced entanglement in the limit of long times. We show that for certain classes of environments the initial state evolves asymptotically to an equilibrium state which is entangled, while for other values of the parameters describing the environment, the entanglement is suppressed and the asymptotic state is separable. The existence of the quantum correlations between the two systems in the asymptotic long-time regime is the result of the competition between entanglement and QD. A summary is given in Sec. 6.

## 2 Markovian master equation for a harmonic oscillator

In the axiomatic formalism based on completely positive quantum dynamical semigroups, the irreversible time evolution of an open system is described by the following general quantum Markovian master equation for the density operator  $\rho(t)$  [11]:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \frac{1}{2\hbar} \sum_j ([V_j \rho(t), V_j^\dagger] + [V_j, \rho(t) V_j^\dagger]). \quad (1)$$

The harmonic oscillator Hamiltonian  $H$  is chosen of the general quadratic form

$$H = H_0 + \frac{\mu}{2}(xp_x + p_x x), \quad H_0 = \frac{1}{2m}p_x^2 + \frac{m\omega^2}{2}x^2 \quad (2)$$

and the operators  $V_j, V_j^\dagger$ , which model the environment, are taken as linear polynomials in coordinate  $x$  and momentum  $p_x$ . Then the master equation (1) takes the following form [12]:

$$\begin{aligned} \frac{d\rho}{dt} = & -\frac{i}{\hbar}[H_0, \rho] - \frac{i}{2\hbar}(\lambda + \mu)[x, \rho p_x + p_x \rho] + \frac{i}{2\hbar}(\lambda - \mu)[p_x, \rho x + x \rho] \\ & - \frac{D_{p_x p_x}}{\hbar^2}[x, [x, \rho]] - \frac{D_{xx}}{\hbar^2}[p_x, [p_x, \rho]] + \frac{D_{xp_x}}{\hbar^2}([x, [p_x, \rho]] + [p_x, [x, \rho]]). \end{aligned} \quad (3)$$

The diffusion coefficients  $D_{xx}, D_{p_x p_x}, D_{xp_x}$  and the dissipation constant  $\lambda$  satisfy the fundamental constraints:  $D_{xx} > 0, D_{p_x p_x} > 0$  and

$$D_{xx}D_{p_x p_x} - D_{xp_x}^2 \geq \frac{\lambda^2 \hbar^2}{4}. \quad (4)$$

In the particular case when the asymptotic state is a Gibbs state  $\rho_G(\infty) = e^{-\frac{H_0}{kT}} / \text{Tr} e^{-\frac{H_0}{kT}}$ , these coefficients become

$$D_{xx} = \frac{\lambda - \mu}{2} \frac{\hbar}{m\omega} \coth \frac{\hbar\omega}{2kT}, \quad D_{p_x p_x} = \frac{\lambda + \mu}{2} \hbar m\omega \coth \frac{\hbar\omega}{2kT}, \quad D_{xp_x} = 0, \quad (5)$$

where  $T$  is the temperature of the thermal bath. In this case, the fundamental constraints are satisfied only if  $\lambda > \mu$  and

$$(\lambda^2 - \mu^2) \coth^2 \frac{\hbar\omega}{2kT} \geq \lambda^2 \quad (6)$$

and the asymptotic values  $\sigma_{xx}(\infty), \sigma_{p_x p_x}(\infty), \sigma_{xp_x}(\infty)$  of the dispersion (variance), respectively correlation (covariance), of the coordinate and momentum, reduce to [12]

$$\sigma_{xx}(\infty) = \frac{\hbar}{2m\omega} \coth \frac{\hbar\omega}{2kT}, \quad \sigma_{p_x p_x}(\infty) = \frac{\hbar m\omega}{2} \coth \frac{\hbar\omega}{2kT}, \quad \sigma_{xp_x}(\infty) = 0. \quad (7)$$

We consider a harmonic oscillator with an initial Gaussian wave function ( $\sigma_x(0)$  and  $\sigma_{p_x}(0)$  are the initial averaged position and momentum of the wave packet)

$$\Psi(x) = \left(\frac{1}{2\pi\sigma_{xx}(0)}\right)^{\frac{1}{4}} \exp\left[-\frac{1}{4\sigma_{xx}(0)}\left(1 - \frac{2i}{\hbar}\sigma_{xp_x}(0)\right)(x - \sigma_x(0))^2 + \frac{i}{\hbar}\sigma_{p_x}(0)x\right], \quad (8)$$

representing a correlated coherent state [13] (squeezed coherent state) with the variances and covariance of coordinate and momentum

$$\sigma_{xx}(0) = \frac{\hbar\delta}{2m\omega}, \quad \sigma_{p_x p_x}(0) = \frac{\hbar m\omega}{2\delta(1 - r^2)}, \quad \sigma_{xp_x}(0) = \frac{\hbar r}{2\sqrt{1 - r^2}}. \quad (9)$$

Here  $\delta$  is the squeezing parameter which measures the spread in the initial Gaussian packet and  $r$ , with  $|r| < 1$  is the correlation coefficient. The initial values (9) correspond to a minimum uncertainty state, since they fulfil the generalized uncertainty relation

$$\sigma_{xx}(0)\sigma_{p_x p_x}(0) - \sigma_{xp_x}^2(0) = \frac{\hbar^2}{4}. \quad (10)$$

For  $\delta = 1$  and  $r = 0$  the correlated coherent state becomes a Glauber coherent state.

From Eq. (3) we derive the evolution equation in coordinate representation:

$$\begin{aligned} \frac{\partial \rho}{\partial t} = & \frac{i\hbar}{2m} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2} \right) \rho - \frac{im\omega^2}{2\hbar} (x^2 - x'^2) \rho \\ & - \frac{1}{2} (\lambda + \mu) (x - x') \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) \rho + \frac{1}{2} (\lambda - \mu) \left[ (x + x') \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right) + 2 \right] \rho \\ & - \frac{D_{p_x p_x}}{\hbar^2} (x - x')^2 \rho + D_{xx} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right)^2 \rho - 2i D_{xp_x} \hbar (x - x') \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right) \rho. \end{aligned} \quad (11)$$

The first two terms on the right-hand side of this equation generate the usual Liou-villian unitary evolution. The third and forth terms are the dissipative terms and have a damping effect (exchange of energy with environment). The last three are noise (diffusive) terms and produce fluctuation effects in the evolution of the system.  $D_{p_x p_x}$  promotes diffusion in momentum and generates decoherence in coordinate  $x$  – it reduces the off-diagonal terms, responsible for correlations between spatially separated pieces of the wave packet. Similarly  $D_{xx}$  promotes diffusion in coordinate and generates decoherence in momentum  $p_x$ . The  $D_{xp_x}$  term is the so-called "anomalous diffusion" term and it does not generate decoherence.

The density matrix solution of Eq. (11) has the general Gaussian form

$$\begin{aligned} \langle x | \rho(t) | x' \rangle = & \left( \frac{1}{2\pi\sigma_{xx}(t)} \right)^{\frac{1}{2}} \exp \left[ -\frac{1}{2\sigma_{xx}(t)} \left( \frac{x + x'}{2} - \sigma_x(t) \right)^2 \right. \\ & \left. - \frac{\sigma(t)}{2\hbar^2\sigma_{xx}(t)} (x - x')^2 + \frac{i\sigma_{xp_x}(t)}{\hbar\sigma_{xx}(t)} \left( \frac{x + x'}{2} - \sigma_x(t) \right) (x - x') + \frac{i}{\hbar} \sigma_{p_x}(t) (x - x') \right], \end{aligned} \quad (12)$$

where

$$\sigma(t) \equiv \sigma_{xx}(t)\sigma_{p_x p_x}(t) - \sigma_{xp_x}^2(t) \quad (13)$$

is the determinant of the covariance matrix

$$\begin{pmatrix} \sigma_{xx}(t) & \sigma_{xp_x}(t) \\ \sigma_{xp_x}(t) & \sigma_{p_x p_x}(t) \end{pmatrix} \quad (14)$$

and represents also the Schrödinger generalized uncertainty function. In the case of a thermal bath we obtain the following stationary state solution for  $t \rightarrow \infty$  ( $\epsilon \equiv \hbar\omega/2kT$ ):

$$\langle x | \rho(\infty) | x' \rangle = \left( \frac{m\omega}{\pi\hbar \coth \epsilon} \right)^{\frac{1}{2}} \exp \left\{ -\frac{m\omega}{4\hbar} \left[ \frac{(x + x')^2}{\coth \epsilon} + (x - x')^2 \coth \epsilon \right] \right\}. \quad (15)$$

### 3 Quantum decoherence

An isolated system has an unitary evolution and the coherence of the state is not lost – pure states evolve in time only to pure states. The QD phenomenon, that is the loss of coherence or the destruction of off-diagonal elements representing coherences between quantum states in the density matrix, can be achieved by introducing an interaction between the system and environment: an initial pure state with a density matrix which contains nonzero off-diagonal terms can non-unitarily evolve into a final mixed state with a diagonal density matrix.

In the literature several measures of the degree of decoherence have been introduced, expressed in terms of the von Neumann entropy [14], linear entropy [15, 16] or the coefficients of the matrix elements of the statistical operator with respect to the energy basis [17].

Using new variables  $\Sigma = (x + x')/2$  and  $\Delta = x - x'$ , the density matrix (12) becomes

$$\rho(\Sigma, \Delta, t) = \sqrt{\frac{\alpha}{\pi}} \exp[-\alpha\Sigma^2 - \gamma\Delta^2 + i\beta\Sigma\Delta + 2\alpha\sigma_x(t)\Sigma + i(\frac{\sigma_{p_x}(t)}{\hbar} - \beta\sigma_x(t))\Delta - \alpha\sigma_x^2(t)], (16)$$

with the abbreviations

$$\alpha = \frac{1}{2\sigma_{xx}(t)}, \quad \gamma = \frac{\sigma(t)}{2\hbar^2\sigma_{xx}(t)}, \quad \beta = \frac{\sigma_{xp_x}(t)}{\hbar\sigma_{xx}(t)}. \quad (17)$$

In the present work we use the representation-independent measure of the degree of QD [8], which is given by the ratio of the dispersion  $1/\sqrt{2\gamma}$  of the off-diagonal element  $\rho(0, \Delta, t)$  to the dispersion  $\sqrt{2/\alpha}$  of the diagonal element  $\rho(\Sigma, 0, t)$  :

$$\delta_{QD}(t) = \frac{1}{2} \sqrt{\frac{\alpha}{\gamma}} = \frac{\hbar}{2\sqrt{\sigma(t)}}. \quad (18)$$

It can easily be shown that  $\delta_{QD}$  is related to the linear entropy [18].

The finite temperature Schrödinger generalized uncertainty function has the expression [19] (with the notation  $\Omega^2 \equiv \omega^2 - \mu^2$ ,  $\omega > \mu$ )

$$\begin{aligned} \sigma(t) = & \frac{\hbar^2}{4} \{ e^{-4\lambda t} [1 - (\delta + \frac{1}{\delta(1-r^2)}) \coth \epsilon + \coth^2 \epsilon] \\ & + e^{-2\lambda t} \coth \epsilon [(\delta + \frac{1}{\delta(1-r^2)} - 2 \coth \epsilon) \frac{\omega^2 - \mu^2 \cos(2\Omega t)}{\Omega^2} \\ & + (\delta - \frac{1}{\delta(1-r^2)}) \frac{\mu \sin(2\Omega t)}{\Omega} + \frac{2r\mu\omega(1 - \cos(2\Omega t))}{\Omega^2 \sqrt{1-r^2}}] + \coth^2 \epsilon \}. \end{aligned} \quad (19)$$

In the limit of long times Eq. (19) yields

$$\sigma(\infty) = \frac{\hbar^2}{4} \coth^2 \epsilon, \quad (20)$$

so that we obtain

$$\delta_{QD}(\infty) = \tanh \frac{\hbar\omega}{2kT}, \quad (21)$$

which for high  $T$  becomes

$$\delta_{QD}(\infty) = \frac{\hbar\omega}{2kT}. \quad (22)$$

To illustrate the dependence on temperature and time of the degree of QD, we represent it in Fig. 1. We see that in general  $\delta_{QD}$  decreases, and therefore QD becomes stronger, with time and temperature, i.e. the density matrix becomes more and more diagonal at higher  $T$  and the contributions of the off-diagonal elements get smaller and smaller. At the same time the degree of purity decreases and the degree of mixedness increases with  $T$ . For  $T = 0$  the asymptotic (final) state is pure and  $\delta_{QD}$  reaches its initial maximum value 1.  $\delta_{QD} = 0$  when the quantum coherence is completely lost, and if  $\delta_{QD} = 1$  there is no QD. Only if  $\delta_{QD} < 1$  we can say that the considered system interacting with the thermal bath manifests QD, when the magnitude of the elements of the density matrix in the position basis are peaked preferentially along the diagonal  $x = x'$ . In Fig. 2 we represent the density matrix in coordinate representation (12) at the initial and final moments of time. The values of density matrix along the diagonal  $x = x'$  represent the probability of finding the system in this position, while off-diagonal values represent the correlations in the density matrix between the points  $x$  and  $x'$ . For simplicity, in Fig. 2 we consider zero values for the initial expectations values of coordinate and momentum, so that the density matrix is centered in origin. Dissipation promotes quantum coherences, whereas fluctuation (diffusion) reduces coherences and promotes QD. The balance of dissipation and fluctuation determines the final equilibrium value of  $\delta_{QD}$  [20].

In order to obtain the expression of the decoherence time, we consider the coefficient  $\gamma$  (17), which measures the contribution of non-diagonal elements in the density matrix (16). For short times ( $\lambda t \ll 1, \Omega t \ll 1$ ), we have:

$$\gamma(t) = -\frac{m\omega}{4\hbar\delta} \left\{ 1 + 2\left[\lambda\left(\delta + \frac{r^2}{\delta(1-r^2)}\right) \coth \epsilon + \mu\left(\delta - \frac{r^2}{\delta(1-r^2)}\right) \coth \epsilon - \lambda - \mu - \frac{\omega r}{\delta\sqrt{1-r^2}}\right]t \right\}. \quad (23)$$

From here we deduce that the quantum coherences in the density matrix decay exponentially and the decoherence time scale is given by

$$t_{deco} = \left\{ 2\left[\lambda\left(\delta + \frac{r^2}{\delta(1-r^2)}\right) \coth \epsilon + \mu\left(\delta - \frac{r^2}{\delta(1-r^2)}\right) \coth \epsilon - \lambda - \mu - \frac{\omega r}{\delta\sqrt{1-r^2}}\right] \right\}^{-1}. \quad (24)$$

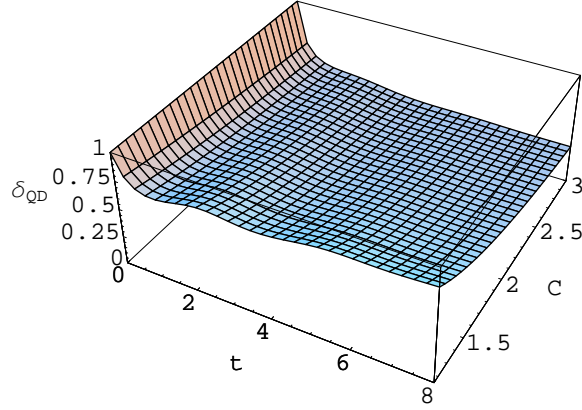


Figure 1: Degree of quantum decoherence  $\delta_{QD}$  versus temperature  $T$  (through  $C \equiv \coth(\hbar\omega/2kT)$ ) and time  $t$  for  $\lambda = 0.2, \mu = 0.1, \delta = 4, r = 0$ .

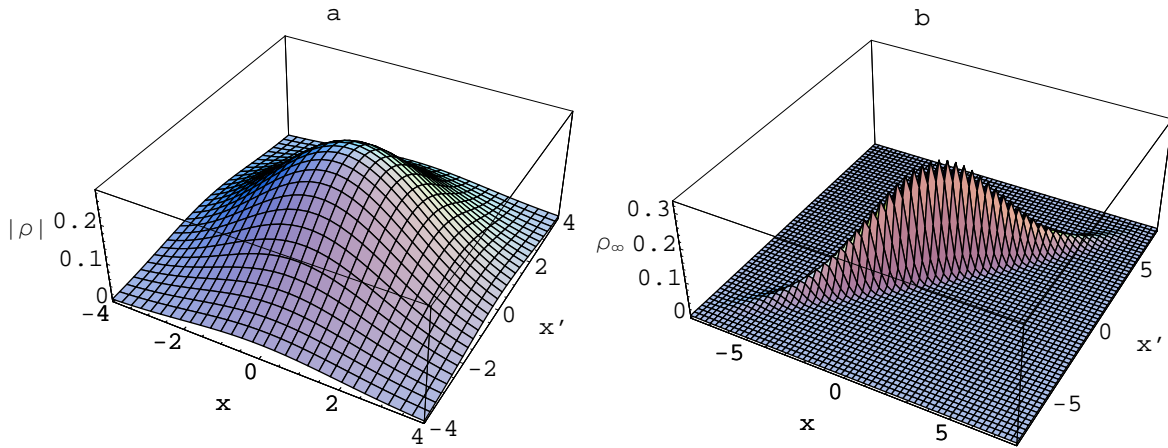


Figure 2: Density matrix  $\rho$  in coordinate representation for  $\lambda = 0.2, \mu = 0.1, \delta = 4, r = 0$ : (a)  $|\rho|$  at the initial time  $t = 0$ ; (b)  $\rho_\infty$  for  $C = 10$ .



The decoherence time depends on the temperature  $T$  and the coupling  $\lambda$  (dissipation coefficient) between the system and environment, the squeezing parameter  $\delta$  and the initial correlation coefficient  $r$ . We notice that the decoherence time is decreasing with increasing dissipation, temperature and squeezing.

For  $r = 0$  we obtain:

$$t_{deco} = \frac{1}{2(\lambda + \mu)(\delta \coth \epsilon - 1)} \quad (25)$$

and at temperature  $T = 0$  [here we have to take  $\mu = 0$  due to Eq. (6)], this becomes

$$t_{deco} = \frac{1}{2\lambda(\delta - 1)}. \quad (26)$$

We see that when the initial state is the usual coherent state ( $\delta = 1$ ), then the decoherence time tends to infinity. This corresponds to the fact that for  $T = 0$  and  $\delta = 1$  the coefficient  $\gamma$  is constant in time, so that the decoherence process does not occur in this case.

At high temperature, expression (24) becomes

$$t_{deco} = \left\{ 2\left[\lambda\left(\delta + \frac{r^2}{\delta(1-r^2)}\right) + \mu\left(\delta - \frac{r^2}{\delta(1-r^2)}\right)\right] \frac{2kT}{\hbar\omega} \right\}^{-1}. \quad (27)$$

If, in addition  $r = 0$ , then we obtain

$$t_{deco} = \frac{\hbar\omega}{4(\lambda + \mu)\delta kT}. \quad (28)$$

The generalized uncertainty function  $\sigma(t)$  (19) has the following behaviour for short times:

$$\sigma(t) = \frac{\hbar^2}{4} \left\{ 1 + 2\left[\lambda\left(\delta + \frac{1}{\delta(1-r^2)}\right) \coth \epsilon + \mu\left(\delta - \frac{1}{\delta(1-r^2)}\right) \coth \epsilon - 2\lambda\right]t \right\}. \quad (29)$$

This expression shows explicitly the contribution for small time of uncertainty that is intrinsic to quantum mechanics, expressed through the Heisenberg uncertainty principle and uncertainty due to the coupling to the thermal environment. From Eq. (29) we can determine the time  $t_d$  when thermal fluctuations become comparable with quantum fluctuations. At high temperature we obtain

$$t_d = \left\{ 2\left[\lambda\left(\delta + \frac{1}{\delta(1-r^2)}\right) + \mu\left(\delta - \frac{1}{\delta(1-r^2)}\right)\right] \frac{2kT}{\hbar\omega} \right\}^{-1}. \quad (30)$$

As expected, the decoherence time  $t_{deco}$  has the same scale as the time  $t_d$  after which thermal fluctuations become comparable with quantum fluctuations [20, 21]. The values of  $t_{deco}$  and  $t_d$  become closer with increasing temperature and squeezing.

When  $t \gg t_{rel}$ , where  $t_{rel} \approx \lambda^{-1}$  is the relaxation time, which governs the rate of energy dissipation, the particle reaches equilibrium with the environment. In the macroscopic domain QD occurs very much faster than relaxation, so that for all macroscopic bodies the dissipation term becomes important much later after the decoherence term has already dominated and diminished the off-diagonal terms of the density matrix. We remark also that  $t_{deco}$  can be of the order of  $t_{rel}$  for sufficiently low temperatures and small wave packet spread (small squeezing coefficient).

## 4 Equations of motion for two independent harmonic oscillators

We are now interested in the generation of entanglement between two harmonic oscillators due to the back-action of the environment on the subsystem. Since the two harmonic oscillators interact with a common environment, there will be induced coupling between the two oscillators even when initially they are uncoupled. Thus, the master equation for the two harmonic oscillators must account for their mutual interaction by their coupling to the environment. We shall study the dynamics of the subsystem composed of two identical non-interacting (independent) oscillators in weak interaction with a large environment, so that their reduced time evolution can be described by a Markovian, completely positive quantum dynamical semigroup, like in the previous case of one harmonic oscillator.

If  $\tilde{\Phi}_t$  is the dynamical semigroup describing the time evolution of the open quantum system in the Heisenberg picture ( $\tilde{\Phi}_t$  is the dual of the dynamical semigroup  $\Phi_t$  which describes the time evolution in the Schrödinger picture given by Eq. (1)), then the master equation is given for an operator  $A$  as follows [11, 12]:

$$\frac{d\tilde{\Phi}_t(A)}{dt} = \frac{i}{\hbar}[H, \tilde{\Phi}_t(A)] + \frac{1}{2\hbar} \sum_j (V_j^\dagger [\tilde{\Phi}_t(A), V_j] + [V_j^\dagger, \tilde{\Phi}_t(A)] V_j). \quad (31)$$

Here, again  $H$  denotes the Hamiltonian of the open quantum system and  $V_j, V_j^\dagger$  which are operators defined on the Hilbert space of  $H$ , represent the interaction of the open system with the environment. Being interested in the set of Gaussian states, we introduce those quantum dynamical semigroups that preserve that set. Therefore  $H$  is taken to be a polynomial of second degree in the coordinates  $x, y$  and momenta  $p_x, p_y$  of the two quantum oscillators and  $V_j, V_j^\dagger$  are taken polynomials of only first degree in these canonical observables. Then in the linear space spanned by the coordinates and

momenta there exist only four linearly independent operators  $V_{j=1,2,3,4}$  [22]:

$$V_j = a_{xj}p_x + a_{yj}p_y + b_{xj}x + b_{yj}y, \quad (32)$$

where  $a_{xj}, a_{yj}, b_{xj}, b_{yj} \in \mathbf{C}$  and

$$V_j^\dagger = a_{xj}^*p_x + a_{yj}^*p_y + b_{xj}^*x + b_{yj}^*y, \quad (33)$$

where  $*$  denotes the complex conjugation. The Hamiltonian  $H$  of the two uncoupled identical harmonic oscillators of mass  $m$  and frequency  $\omega$  is given by

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{m\omega^2}{2}(x^2 + y^2). \quad (34)$$

The fact that  $\tilde{\Phi}_t$  is a dynamical semigroup implies the positivity of the following matrix formed by the scalar products of the four vectors  $\mathbf{a}_x, \mathbf{a}_y, \mathbf{b}_x, \mathbf{b}_y$  whose entries are the components  $a_{xj}, a_{yj}, b_{xj}, b_{yj}$ , respectively:

$$\frac{1}{2}\hbar \begin{pmatrix} (\mathbf{a}_x \mathbf{a}_x) & (\mathbf{a}_x \mathbf{b}_x) & (\mathbf{a}_x \mathbf{a}_y) & (\mathbf{a}_x \mathbf{b}_y) \\ (\mathbf{b}_x \mathbf{a}_x) & (\mathbf{b}_x \mathbf{b}_x) & (\mathbf{b}_x \mathbf{a}_y) & (\mathbf{b}_x \mathbf{b}_y) \\ (\mathbf{a}_y \mathbf{a}_x) & (\mathbf{a}_y \mathbf{b}_x) & (\mathbf{a}_y \mathbf{a}_y) & (\mathbf{a}_y \mathbf{b}_y) \\ (\mathbf{b}_y \mathbf{a}_x) & (\mathbf{b}_y \mathbf{b}_x) & (\mathbf{b}_y \mathbf{a}_y) & (\mathbf{b}_y \mathbf{b}_y) \end{pmatrix}. \quad (35)$$

For simplicity we take this matrix of the following form, where all coefficients  $D_{xx}, D_{xp_x}, \dots$  and  $\lambda$  are real quantities:

$$\begin{pmatrix} D_{xx} & -D_{xp_x} - i\hbar\lambda/2 & D_{xy} & -D_{xp_y} \\ -D_{xp_x} + i\hbar\lambda/2 & D_{p_x p_x} & -D_{yp_x} & D_{p_x p_y} \\ D_{xy} & -D_{yp_x} & D_{yy} & -D_{yp_y} - i\hbar\lambda/2 \\ -D_{xp_y} & D_{p_x p_y} & -D_{yp_y} + i\hbar\lambda/2 & D_{p_y p_y} \end{pmatrix}. \quad (36)$$

It follows that the principal minors of this matrix are positive or zero. From the Cauchy-Schwarz inequality the following relations for the coefficients defined in Eq. (36) hold (from now on we put, for simplicity,  $\hbar = 1$ ):

$$D_{xx}D_{yy} - D_{xy}^2 \geq 0, \quad D_{xx}D_{p_x p_x} - D_{xp_x}^2 \geq \frac{\lambda^2}{4}, \quad D_{xx}D_{p_y p_y} - D_{xp_y}^2 \geq 0, \quad (37)$$

$$D_{yy}D_{p_x p_x} - D_{yp_x}^2 \geq 0, \quad D_{yy}D_{p_y p_y} - D_{yp_y}^2 \geq \frac{\lambda^2}{4}, \quad D_{p_x p_x}D_{p_y p_y} - D_{p_x p_y}^2 \geq 0. \quad (38)$$

The matrix of the coefficients (36) can be conveniently written as

$$\begin{pmatrix} C_1 & C_3 \\ C_3^\dagger & C_2 \end{pmatrix}, \quad (39)$$

in terms of  $2 \times 2$  matrices  $C_1 = C_1^\dagger$ ,  $C_2 = C_2^\dagger$  and  $C_3$ . This decomposition has a direct physical interpretation: the elements containing the diagonal contributions

$C_1$  and  $C_2$  represent diffusion and dissipation coefficients corresponding to the first, respectively the second, system in absence of the other, while the elements in  $C_3$  represent environment generated couplings between the two, initially independent, oscillators.

The variance and covariance of self-adjoint operators  $A_1$  and  $A_2$  can be written with the density operator  $\rho$ , describing the initial state of the quantum system, as follows:

$$\sigma_{A_1 A_2}(t) = \frac{1}{2} \text{Tr}(\rho \tilde{\Phi}_t(A_1 A_2 + A_2 A_1)). \quad (40)$$

We introduce the following  $4 \times 4$  covariance matrix:

$$\sigma(t) = \begin{pmatrix} \sigma_{xx} & \sigma_{xp_x} & \sigma_{xy} & \sigma_{xp_y} \\ \sigma_{xp_x} & \sigma_{p_x p_x} & \sigma_{yp_x} & \sigma_{p_x p_y} \\ \sigma_{xy} & \sigma_{yp_x} & \sigma_{yy} & \sigma_{yp_y} \\ \sigma_{xp_y} & \sigma_{p_x p_y} & \sigma_{yp_y} & \sigma_{p_y p_y} \end{pmatrix}. \quad (41)$$

By direct calculation we obtain [22]:

$$\frac{d\sigma}{dt} = Y\sigma + \sigma Y^T + 2D, \quad (42)$$

where

$$Y = \begin{pmatrix} -\lambda & 1/m & 0 & 0 \\ -m\omega^2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1/m \\ 0 & 0 & -m\omega^2 & -\lambda \end{pmatrix}, \quad (43)$$

$D$  is the matrix of the diffusion coefficients

$$D = \begin{pmatrix} D_{xx} & D_{xp_x} & D_{xy} & D_{xp_y} \\ D_{xp_x} & D_{p_x p_x} & D_{yp_x} & D_{p_x p_y} \\ D_{xy} & D_{yp_x} & D_{yy} & D_{yp_y} \\ D_{xp_y} & D_{p_x p_y} & D_{yp_y} & D_{p_y p_y} \end{pmatrix} \quad (44)$$

and  $Y^T$  is the transposed matrix of  $Y$ . The time-dependent solution of Eq. (42) is given by [22]

$$\sigma(t) = M(t)(\sigma(0) - \sigma(\infty))M^T(t) + \sigma(\infty), \quad (45)$$

where  $M(t) = \exp(tY)$ . The matrix  $M(t)$  has to fulfil the condition  $\lim_{t \rightarrow \infty} M(t) = 0$ . In order that this limit exists,  $Y$  must only have eigenvalues with negative real parts. The values at infinity are obtained from the equation [22]

$$Y\sigma(\infty) + \sigma(\infty)Y^T = -2D. \quad (46)$$

## 5 Environment induced entanglement

The two-mode Gaussian state is entirely specified by its covariance matrix  $\sigma$  (41), which is a real, symmetric and positive matrix with the following block structure:

$$\sigma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \quad (47)$$

where  $A$ ,  $B$  and  $C$  are  $2 \times 2$  matrices. Their entries are correlations of the canonical operators  $x, y, p_x, p_y$ ,  $A$  and  $B$  denote the symmetric covariance matrices for the individual reduced one-mode states, while the matrix  $C$  contains the cross-correlations between modes. The entries of the covariance matrix depend on  $Y$  and  $D$  and can be calculated from Eq. (46). To simplify further the calculations, we shall consider environments for which the two diagonal submatrices in Eq. (39) are equal:  $C_1 = C_2$ , so that  $D_{xx} = D_{yy}$ ,  $D_{xp_x} = D_{yp_y}$ ,  $D_{p_x p_x} = D_{p_y p_y}$ . In addition, in the matrix  $C_3$  we take  $D_{xp_y} = D_{yp_x}$ . Then both unimodal covariance matrices are equal,  $A = B$  and the entanglement matrix  $C$  is symmetric. With the chosen coefficients, we obtain the following elements of the asymptotic entanglement matrix  $C$ :

$$\sigma_{xy}(\infty) = \frac{m^2(2\lambda^2 + \omega^2)D_{xy} + 2m\lambda D_{xp_y} + D_{p_x p_y}}{2m^2\lambda(\lambda^2 + \omega^2)}, \quad (48)$$

$$\sigma_{xp_y}(\infty) = \sigma_{yp_x}(\infty) = \frac{-m^2\omega^2 D_{xy} + 2m\lambda D_{xp_y} + D_{p_x p_y}}{2m(\lambda^2 + \omega^2)}, \quad (49)$$

$$\sigma_{p_x p_y}(\infty) = \frac{m^2\omega^4 D_{xy} - 2m\omega^2\lambda D_{xp_y} + (2\lambda^2 + \omega^2)D_{p_x p_y}}{2\lambda(\lambda^2 + \omega^2)} \quad (50)$$

and of the matrices  $A$  and  $B$ :

$$\sigma_{xx}(\infty) = \sigma_{yy}(\infty) = \frac{m^2(2\lambda^2 + \omega^2)D_{xx} + 2m\lambda D_{xp_x} + D_{p_x p_x}}{2m^2\lambda(\lambda^2 + \omega^2)}, \quad (51)$$

$$\sigma_{xp_x}(\infty) = \sigma_{yp_y}(\infty) = \frac{-m^2\omega^2 D_{xx} + 2m\lambda D_{xp_x} + D_{p_x p_x}}{2m(\lambda^2 + \omega^2)}, \quad (52)$$

$$\sigma_{p_x p_x}(\infty) = \sigma_{p_y p_y}(\infty) = \frac{m^2\omega^4 D_{xx} - 2m\omega^2\lambda D_{xp_x} + (2\lambda^2 + \omega^2)D_{p_x p_x}}{2\lambda(\lambda^2 + \omega^2)}. \quad (53)$$

With these quantities we calculate the determinant of the entanglement matrix:

$$\det C = \frac{1}{4\lambda^2(\lambda^2 + \omega^2)}[(m\omega^2 D_{xy} + \frac{1}{m}D_{p_x p_y})^2 + 4\lambda^2(D_{xy}D_{p_x p_y} - D_{xp_y}^2)]. \quad (54)$$

It is very interesting that the general theory of open quantum systems allows couplings via the environment between uncoupled oscillators. According to the definitions of the environment parameters, the diffusion coefficients above can be different from zero and can simulate an interaction between the uncoupled oscillators. Indeed, the Gaussian states with  $\det C \geq 0$  are separable states, but for  $\det C < 0$ , it may be possible that the asymptotic equilibrium states are entangled, as it will be shown in the following.

On general grounds, one expects that the effects of decoherence, counteracting entanglement production, be dominant in the long-time regime, so that no quantum correlation (entanglement) is expected to be left at infinity. Nevertheless, there are situations in which the environment allows the presence of entangled asymptotic equilibrium states. In order to investigate whether an external environment can actually entangle the two independent systems, we can use the partial transposition criterion [9, 10]: a state results entangled if and only if the operation of partial transposition does not preserve its positivity. Simon [10] obtained the following necessary and sufficient criterion for separability:  $S \geq 0$ , where

$$S \equiv \det A \det B + \left(\frac{1}{4} - |\det C|\right)^2 - \text{Tr}[AJCJB JC^T J] - \frac{1}{4}(\det A + \det B) \quad (55)$$

and  $J$  is the  $2 \times 2$  symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (56)$$

In order to analyze the possible persistence of the environment induced entanglement in the asymptotic long-time regime, we consider the environment characterized by the following values of its parameters:  $m^2\omega^2 D_{xx} = D_{p_x p_x}$ ,  $D_{x p_x} = 0$ ,  $m^2\omega^2 D_{xy} = D_{p_x p_y}$ . In this case the Simon expression (55) takes the form:

$$S = \left( \frac{m^2\omega^2(D_{xx}^2 - D_{xy}^2)}{\lambda^2} + \frac{D_{x p_y}^2}{\lambda^2 + \omega^2} - \frac{1}{4} \right)^2 - 4 \frac{m^2\omega^2 D_{xx}^2 D_{x p_y}^2}{\lambda^2(\lambda^2 + \omega^2)}. \quad (57)$$

For environments characterized by such coefficients that the expression (57) is negative, the asymptotic final state is entangled. In particular, if  $D_{xy} = 0$ , we obtain that  $S < 0$ , i.e. the asymptotic final state is entangled, for the following range of values of the coefficient  $D_{x p_y}$  characterizing the environment:

$$\frac{m\omega D_{xx}}{\lambda} - \frac{1}{2} < \frac{D_{x p_y}}{\sqrt{\lambda^2 + \omega^2}} < \frac{m\omega D_{xx}}{\lambda} + \frac{1}{2}, \quad (58)$$

where the coefficient  $D_{xx}$  satisfies the condition  $m\omega D_{xx}/\lambda \geq 1/2$ , equivalent with the unimodal uncertainty relation. If the coefficients do not fulfil the inequalities (58), then  $S \geq 0$  and therefore the asymptotic final state of the considered bipartite system is separable.

## 6 Summary

We have studied QD with the Markovian equation of Lindblad for a system consisting of an one-dimensional harmonic oscillator in interaction with a thermal bath in the framework of the theory of open quantum systems based on completely positive quantum dynamical semigroups. In the same framework we investigated the existence of the asymptotic quantum entanglement for a subsystem composed of two uncoupled identical harmonic oscillators interacting with an environment.

(1) We have shown that QD in general increases with time and temperature. For large temperatures, QD is strong and the degree of mixedness is high, while for zero temperature the asymptotic final state is pure. With increasing squeezing parameter and initial correlation, QD becomes stronger, but the asymptotic value of the degree of QD does not depend on the initial squeezing and correlation, it depends on temperature only. QD is expressed by the loss of quantum coherences in the case of a thermal bath at finite temperature.

(2) We determined the general expression of the decoherence time, which shows that it is decreasing with increasing dissipation, temperature and squeezing. We have also shown that the decoherence time has the same scale as the time after which thermal fluctuations become comparable with quantum fluctuations and the values of these scales become closer with increasing temperature and squeezing.

(3) By using the Peres-Simon necessary and sufficient condition for separability of two-mode Gaussian states, we have shown that for certain classes of environments the initial state evolves asymptotically to an equilibrium state which is entangled, i.e. there exist non-local quantum correlations for the bipartite states of the two harmonic oscillator subsystem, while for other values of the coefficients describing the environment, the asymptotic state is separable.

The obtained results can represent a useful basis for the description of the connection between uncertainty, decoherence and correlations (entanglement) of open quantum systems with their environment. Due to the increasing interest manifested towards the continuous variables approach [23] to the theory of quantum information, these results, in particular the possibility of maintaining a bipartite entanglement in a diffusive-dissipative environment even for asymptotic long times, could be useful for both phenomenological and experimental applications in the field of quantum information processing and communication.

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